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Aggregation Dynamics in a Self-Gravitating One-Dimensional Gas

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Aggregation of mass by perfectly inelastic collisions in a one-dimensional self-gravitating gas is studied. The binary collisions are subject to the laws of mass and momentum conservation. A method to obtain an exact probabilistic description of aggregation is presented. Since the one-dimensional gravitational attraction is confining, all particles will eventually form a single body. The detailed analysis of the probability $P_n(t)$ of such a complete merging before time t is performed for initial states of n equidistant identical particles with uncorrelated velocities. It is found that for a macroscopic amount of matter ($n \to \infty$), this probability vanishes before a characteristic time t^*. In the limit of a continuous initial mass distribution the exact analytic form of $P_n(t)$ is derived. The analysis of collisions leading to the time-variation of $P_n(t)$ reveals that in fact the merging into macroscopic bodies always occurs in the immediate vicinity of t^*. For $t > t^*$, and n large, $P_n(t)$ describes events corresponding to the final aggregation of remaining microscopic fragments.

KEY WORDS: Inelastic collisions; gravitational forces; aggregation of mass.

1. INTRODUCTION

Our object in this paper is to study the dynamics of a one-dimensional gas composed of n point particles coupled by gravitational interaction, and aggregating via perfectly inelastic collisions. The aggregation process leads eventually to the creation of a single mass out of the initial dust of particles. Although the method is similar to that used in the study of ballistic aggregation[1], the physics of formation of massive aggregates in the presence of gravity differs substantially from the ballistic case by the

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appearance of a finite time scale characterizing the gravitation-dominated regime.

The potential energy of a pair of particles occupying positions x_i and x_j equals

$$\gamma m_i m_j |x_i - x_j| \tag{1}$$

where m_i and m_j are their masses, and γ denotes the gravitational constant. The motion of the system between collisions corresponds thus to the Hamiltonian

$$\sum_{i=1}^{n} \frac{p_i^2}{2m_i} + \gamma \sum_{i<j}^{n} \sum^{n} m_i m_j |x_i - x_j| \tag{2}$$

where p_i is the momentum of particle i.

When two particles collide they merge instantaneously forming a single point mass. These perfectly inelastic collisions are subject to the momentum and mass conservation laws. The particle formed out of particles i and j thus has the mass $(m_i + m_j)$, and the momentum $(p_i + p_j)$.

In one dimension only the nearest neighbors can collide. Given the initial configuration

$$x_1 < x_2 < \cdots < x_n \tag{3}$$

the equations of motion of the neighboring pair $(i, i+1)$ corresponding to the Hamiltonian (2) read

$$\begin{aligned} \frac{d^2 x_i}{dt^2} &= \gamma(m_{i+1} + \Delta M_{(i,i+1)}) \\ \frac{d^2 x_{i+1}}{dt^2} &= \gamma(\Delta M_{(i,i+1)} - m_i) \end{aligned} \tag{4}$$

where

$$\Delta M_{(i,i+1)} = \sum_{j=i+2}^{n} m_j - \sum_{j=1}^{i-1} m_j$$

is the difference between the mass of the system to the right and to the left of the pair $(i, i+1)$.

The fundamental remark about the aggregation dynamics is that when an inelastic collision between particles i and $i+1$ occurs the newly formed mass $(m_i + m_j)$ follows the trajectory of their center of mass. Indeed, this trajectory satisfies the equation

$$\frac{d^2}{dt^2}\left(\frac{m_i x_i + m_{i+1} x_{i+1}}{m_i + m_{i+1}}\right) = \gamma \, \Delta M_{(i, i+1)} \tag{5}$$

which is precisely the equation of motion of the aggregate $(m_i + m_{i+1})$. As at the moment of formation the new particle is on the trajectory of the center of mass and acquires instantaneously its momentum, its further motion reduces to the continuation of the center of mass displacement. This remarkable property of the gravitational interaction (1) enables us to extend the method developed by us in the study of ballistic aggregation to the qualitatively different case of gravitational forces.

In this paper we shall be mainly concerned with information which can be obtained from the knowledge of the probability $P_n(t)$ to find at time t a single mass $M = m_1 + m_2 + \cdots + m_n$. However, the approach used to derive a microscopic formula for $P_n(t)$ is general, and can be applied to study other quantities relevant to the dynamics of the aggregating gas.

In order to derive a formula for $P_n(t)$, let us consider the partition of the whole system into two adjacent clusters containing particles $\{1, 2,..., r\}$ and $\{r+1,..., n\}$. The centers of mass of the clusters occupy at $t = 0$ some points $X^r(0)$ and $X^{n-r}(0)$ with

$$X^r(0) < X^{n-r}(0)$$

[see Eq. (3)]. The necessary and sufficient condition for merging of the whole system into a single mass within the time interval $(0, t]$ can be formulated as the requirement that for any partition into r and $(n-r)$ particles the centers of mass of the two clusters cross before the moment t

$$X^r(t) \geqslant X^{n-r}(t), \qquad r = 1, 2,..., n-1 \tag{6}$$

The reasoning here is identical to that applied to the ballistic case. It consists in realizing that the aggregates move along the center-of-mass trajectories, and thus the presence at time t of two (or more) masses is equivalent to an unperturbed motion of two (or more) center-of-mass trajectories.

We conclude that the probability of a complete aggregation of the gas into a single massive particle before time t is given by the formula

$$P_n(t) = \left\langle \prod_{r=1}^{n-1} \theta(X^r(t) - X^{n-r}(t)) \right\rangle \tag{7}$$

where θ is a unit step function, and the brackets $\langle \cdots \rangle$ denote the average over the distribution of initial masses (the state of the system at $t = 0$).

2. INITIAL STATE

We shall suppose here that at the initial moment $t=0$ the interacting gas is composed of identical particles of mass m equally spaced at distance a on the line

$$x_j = ja, \qquad j = 1, 2, ..., n \tag{8}$$

In this case the centers of mass of the clusters $\{1, ..., r\}$ and $\{r+1, ..., n\}$ occupy points

$$X^r(0) = (r+1)\frac{a}{2}$$

and

$$X^{n-r}(0) = (n+r+1)\frac{a}{2}$$

respectively.

Using the explicit formulas

$$\begin{aligned} X^r(t) &= (r+1)\frac{a}{2} + \frac{1}{r}\left(\sum_{j=1}^{r} v_j\right) t + \gamma(n-r)\, m\frac{t^2}{2} \\ X^{n-r}(t) &= (n+r+1)\frac{a}{2} + \frac{1}{n-r}\left(\sum_{j=r+1}^{n} v_j\right) t - \gamma r m \frac{t^2}{2} \end{aligned} \tag{9}$$

we thus find that the characteristic function corresponding to inequalities (6) has the form

$$\begin{aligned} &\prod_{r=1}^{n-1} \theta\{X^r(t) - X^{n-r}(t)\} \\ &\qquad = \prod_{r=1}^{n-1} \theta\left\{\sum_{j=1}^{r} v_j - \frac{r}{n}\sum_{j=1}^{n} v_j + m\tau\frac{r(n-r)}{2}\right\} \end{aligned} \tag{10}$$

where

$$\rho = \frac{m}{a}$$

is the initial mass density of the system and the variable τ is related to time t by

$$\tau = \gamma t - \frac{1}{\rho t} \tag{11}$$

We shall study the case where in the initial state no correlations between the velocities of the particles are present, and each particle has the same velocity distribution with a symmetric probability density $\phi(v) = \phi(-v)$. Then, the probability (7) of formation of a single mass $M = nm$, written now as a function of τ, takes the form

$$P_n(\tau) = \int dv_1 \int dv_2 \cdots \int dv_n \, \phi(v_1)\, \phi(v_2) \cdots \phi(v_n) \times \prod_{r=1}^{n-1} \theta\left\{ \sum_{j=1}^{r} v_j - \frac{r}{n} \sum_{j=1}^{n} v_j + m\tau \frac{r(n-r)}{2} \right\} \tag{12}$$

Performing the change of the integration variables

$$V_r = v_1 + v_2 + \cdots + v_r, \qquad r = 1, 2, ..., n \tag{13}$$

we eventually find

$$P_n(\tau) = \int dV_1 \cdots \int dV_n \, \phi(V_1)\, \phi(V_2 - V_1) \cdots \phi(V_n - V_{n-1}) \times \prod_{r=1}^{n-1} \theta\left\{ V_r - \frac{r}{n} V_n + m\tau \frac{r(n-r)}{2} \right\} \tag{14}$$

Equations (10) and (14) reveal the existence of a characteristic time related to the gravitational attraction. This is $\tau = 0$, or

$$t = t^* = \frac{1}{\sqrt{\gamma\rho}} \tag{15}$$

It turns out that the time t^* plays an essential role in the dynamics of aggregation.

3. CHARACTERISTIC TIME

The relevance of the time scale related to t^* is most clearly seen when one considers the case of the initial cloud of particles being at rest

$$\phi(u) = \delta(u) \tag{16}$$

(δ denotes the Dirac distribution). The further evolution is entirely induced by gravitational attraction. Eq. (14) reduces then to the simple formula

$$P_n(\tau) = \theta(\tau) = \theta(t - t^*) \tag{17}$$

When $t < t^*$ no collision takes place. Then, all particles collide simultaneously, precisely at the moment t^*. For $t > t^*$ the system is transformed into a single mass with probability 1. One can expect in general important changes in the structure of the gravitating gas (macroscopic aggregation) to take place in the vicinity of t^*.

In order to further investigate the role of this time scale let us analyze the case of the velocity distribution with $\phi(u)$ vanishing outside a finite interval $[-v_{\max}, v_{\max}]$. The argument of the θ function in Eq. (11) has the form

$$f_r(v_1,..., v_n) + m\tau \frac{r(n-r)}{2}$$

where

$$f_r(v_1,..., v_n) = \sum_{j=1}^{r} v_j - \frac{r}{n} \sum_{j=1}^{n} v_j \tag{18}$$

When

$$v_j \in [-v_{\max}, v_{\max}], \qquad j = 1, 2,..., n$$

the minimal and the maximal values of the function $f_r(v_1,..., v_n)$ equal $-2v_{\max} r(n-r)/n$ and $+2v_{\max} r(n-r)/n$, respectively. It follows that when

$$m\tau \frac{r(n-r)}{2} < -2v_{\max} \frac{r(n-r)}{n}$$

the complete aggregation cannot take place, whereas for

$$m\tau \frac{r(n-r)}{2} > 2v_{\max} \frac{r(n-r)}{n}$$

the formation of a single mass nm must have occurred. We thus conclude that

$$P_n(\tau) = \begin{cases} 1 & \text{if} \quad \tau > 4v_{\max}/nm \\ 0 & \text{if} \quad \tau < -4v_{\max}/nm \end{cases} \tag{19}$$

We observe from (2) that typical kinetic and potential energies in the system are of the order $E_{\text{kin}} = nmv_{\max}^2$ and $E_{\text{pot}} = \gamma n^3 m^2 a$, repectively. When expressed in terms of variable t, Eq. (19) implies that the complete aggregation is impossible when

$$\frac{t}{t^*} < -2\sqrt{\frac{E_{\text{kin}}}{E_{\text{pot}}}} + \sqrt{1 + 4\frac{E_{\text{kin}}}{E_{\text{pot}}}} \tag{20}$$

but it occurs with certainty if

$$\frac{t}{t^*} > 2\sqrt{\frac{E_{\text{kin}}}{E_{\text{pot}}}} + \sqrt{1 + 4\frac{E_{\text{kin}}}{E_{\text{pot}}}} \tag{21}$$

These inequalities are entirely expressed in terms of the dimensionless parameter $E_{\text{kin}}/E_{\text{pot}} = v_{\max}^2 \rho/n^2 m^2 \gamma$. One sees that the probability $P_n(\tau)$ becomes a step function in the gravity-dominated regime $E_{\text{kin}} \ll E_{\text{pot}}$. In particular, when $v_{\max} \to 0$, the result (17) is recovered.

It is to be noticed at this point that the ideally inelastic collisions always reduce the kinetic energy (the amount of energy lost in a binary collision equals the kinetic energy of the reduced mass moving with the relative velocity). It follows that at the initial stage of the evolution $v_{\max}$ gets reduced which favors the manifestation of the gravitation dominated dynamics. In the general case, Eq. (19) locates in a precise way the time interval in which the evolution of the probability $P_n(\tau)$ is taking place.

4. GRAVITATIONAL COLLAPSE WITH GAUSSIAN INITIAL VELOCITY DISTRIBUTION

To proceed further in a more detailed analytic study of $P_n(\tau)$ we assume from now on that the initial velocity distribution is Gaussian with the mean-square fluctuation λ

$$\phi_\lambda(v) = \frac{1}{\sqrt{2\pi}\,\lambda} \exp\left(-\frac{v^2}{2\lambda^2}\right) \tag{22}$$

The inequalities (19) cannot be applied directly in this case because of a nonvanishing probability to find particles with arbitrarily big kinetic energy. The change of the integration variables in Eq. (14)

$$\begin{aligned} u_r &= \frac{1}{\lambda}\left(V_r - \frac{r}{n} V_n\right), \qquad r = 1, 2, \ldots, n-1 \\ u_n &= \frac{V_n}{\lambda} \end{aligned} \tag{23}$$

yields the expression

$$P_n(\tau)=\sqrt{2\pi n}\int du_1\cdots\int du_{n-1}$$

$$\times\,\phi(u_1)\,\phi(u_2-u_1)\cdots\phi(u_{n-1}-u_{n-2})\,\phi(-u_{n-1})$$

$$\times\prod_{r=1}^{n-1}\theta\left\{u_r+\frac{m\tau}{2\lambda}r(n-r)\right\} \tag{24}$$

In Eq. (24), $\phi(u)$ is the Gaussian distribution (22) with covariance $\lambda=1$ and the factor $(2\pi n)^{1/2}$ results from the integration over the total velocity distribution $\exp(-u_n^2/2n)$, which factorizes out after transformation (23).

At this point, it is useful to observe that the velocities $u_r=u(s_r)$ can be considered as the restriction of a continuous Brownian process $u(s)$ to integer "times" $s_r=r$, where $r=1,\dots,n-1$. With this interpretation

$$P_n(\tau)=\sqrt{2\pi n}\,E_{\mathrm{W}}\left[u(s_r)\geqslant-\frac{m\tau}{2\lambda}s_r(n-s_r),\ r=1,\dots,n-1\,\middle|\,u(n)=0\right] \tag{25}$$

is [up to the normalizing factor $(2\pi n)^{1/2}$] the conditional Wiener expectation for the paths starting and ending in the origin within "time" $s_n=n$, and being above the parabolic barrier

$$\frac{-m\tau}{2\lambda}s(n-s)$$

at all integer "times" $s_r=r$, for $r=1,\dots,n-1$.

For $\tau<0$ (or $t<t^*$) the paths have to overcome the positive barriers $u(s_r)\geqslant m\,|\tau|\,s_r(n-s_r)/2\lambda$. Then an upperbound to $P_n(\tau)$ can easily be given by keeping only the requirement that the paths have to pass beyond the highest point $m|\tau|n^2/8\lambda$ (assuming n even). This gives

$$P_n(\tau)\leqslant\sqrt{2\pi n}\int_{m\,|\tau|\,n^2/8\lambda}^{\infty}dy\exp\left(-\frac{2y^2}{n}\right)$$

$$\simeq\frac{\lambda}{m\,|\tau|}\sqrt{\frac{8\pi}{n}}\exp\left[-\left(\frac{\tau}{\tau_-}\right)^2\right] \tag{26}$$

as $\tau\to-\infty$, with characteristic time

$$\tau_-=\frac{4\lambda\sqrt{2}}{m}\frac{1}{n^{3/2}} \tag{27}$$

For $\tau > 0$ (or $t > t^*$) we expand the product of step functions in (24) as

$$\prod_{r=1}^{n-1}\left[1-\theta\left(-u_r-\frac{m\tau}{2\lambda}r(n-r)\right)\right]$$
$$=1-\sum_{r=1}^{n-1}\theta\left(-u_r-\frac{m\tau}{2\lambda}r(n-r)\right)+\cdots \tag{28}$$

where the terms indicated by an ellipsis involve sums of products of at least two θ-functions. In the limit $\tau \to \infty$, the dominant contribution will be given by the two terms $r=1$ and $r=n-1$ in the sum (28), corresponding now to the weakest constraint. Thus

$$P_n(\tau) \simeq 1-2\sqrt{\frac{n}{n-1}}\int_{m\tau(n-1)/2\lambda}^{\infty} dy \exp\left[-\frac{ny^2}{2(n-1)}\right]$$
$$\simeq 1-\frac{\lambda}{m\tau}\frac{4}{\sqrt{n(n-1)}}\exp\left[-\left(\frac{\tau}{\tau_+}\right)^2\right] \tag{29}$$

as $\tau \to \infty$, with

$$\tau_+ = \frac{2\lambda\sqrt{2}}{m}\frac{1}{\sqrt{n(n-1)}} \tag{30}$$

Although the asymptotic behaviors of $P_n(\tau)$ as $\tau \to \pm\infty$ are both Gaussian, there is a noticeable asymmetry. Comparing (27) to (30) one sees that the ratio of the characteristic times τ_+/τ_- is of the order $\sqrt{n}$. In particular, if the total mass $M = nm$ to be formed is fixed, but n is very large, τ_- vanishes as $(M\sqrt{n})^{-1}$, whereas τ_+ remains of the order M^{-1} (this asymmetry in the behavior of P_n for $t < t^*$ and $t > t^*$ appears clearly in the limit of a continuous initial mass distribution discussed in the next section).

At $\tau = 0$ (i.e., $t = t^*$), $P_n(0)$ represents the normalized measure of closed Brownian paths $u(0) = u(n) = 0$ that take positive values at all integer times $s_r = r$, $r = 1,\ldots, n-1$. It is shown in Appendix A that

$$P_n(0) = \frac{1}{n} \tag{31}$$

Thus the fraction of events for which the total mass $M = nm$ is formed before t^* is always equal to $1/n$. This value is universal in the sense that it does not depend on other parameters (in particular on the velocity dispersion λ, as long as $\lambda > 0$). For $n \to \infty$, the probability $P_n(0) \to 0$. As $P_n(\tau)$ is increasing with τ, this implies vanishing of $P_n(\tau)$ for all $\tau < 0$: the complete aggregation of a macroscopic mass becomes impossible for $t < t^*$.

Finally, we note that the step functions in (24) tend to $\theta(\tau)$ if either $\lambda \to 0$ or $n \to \infty$. This implies (by dominated convergence) the limits

$$\lim_{\lambda \to 0} P_n(\tau) = \theta(\tau) \tag{32}$$

$$\lim_{n \to \infty} P_n(\tau) = \theta(\tau) \tag{33}$$

Both limits correspond to the gravitation-dominated regime, and the conclusions are the same as those following from the bound (19) valid for densities $\phi(v)$ with a compact support.

On the other hand, we see that

$$\lim_{\lambda \to \infty} P_n(\tau) = P_n(0) = \frac{1}{n} \tag{34}$$

as a consequence of (31). The limit (34) describes the situation dominated by the kinetic energy fluctuations at any time only the fraction $1/n$ of events can lead to the aggregation of the total mass.

5. CONTINUUM LIMIT

We consider now a limiting situation in which a proper balance between the kinetic and gravitational energies leads to a nontrivial form of the probability $P_n(\tau)$. This is achieved by letting $n \to \infty$ and $m \to 0$ in such a way that the total mass $M = nm$, the initial mass density $\rho = m/a$, as well as λ are kept constant. As in this limit $a \to 0$, the initial mass distribution becomes uniform in an interval of length M/ρ. Equivalently, one can keep m and a fixed, but let $n \to \infty$ and $\lambda \to \infty$, with n/λ = const. We thus rewrite $P_n(\tau)$ in the form

$$\begin{aligned} P_n(\tau) = {} & \sqrt{2\pi n} \int du_1 \cdots \int du_{n-1} \\ & \times \phi(u_1)\, \phi(u_2 - u_1) \cdots \phi(u_{n-1} - u_{n-2})\, \phi(-u_{n-1}) \\ & \times \prod_{r=1}^{n-1} \theta\{u_r + h_n(r)\} \\ = {} & \sqrt{2\pi n}\, E_{\mathrm{W}}[u(r) \geqslant -h_n(r), r = 1,..., n-1 \mid u(n) = 0] \end{aligned} \tag{35}$$

where the constraint $h_n(r)$ is given by

$$h_n(r) = \alpha \tau r \left(1 - \frac{r}{n}\right), \qquad \alpha = \frac{nm}{2\lambda} \tag{36}$$

Then we find the following result.

Proposition. For α fixed, the probabilities $P_n(\tau)$ converge to a limit $P(\tau)$ as $n \to \infty$, with

$$P(\tau) = \begin{cases} 0 & \text{if} \quad \tau \leqslant 0 \\ \exp\left[-\sqrt{\frac{2}{\pi}} \int_{\alpha\tau}^{\infty} dy\, F_{1/2}\left(\frac{y^2}{2}\right)\right] & \text{if} \quad \tau > 0 \end{cases} \tag{37}$$

where

$$F_\sigma(y) = \sum_{k=1}^{\infty} \frac{\exp(-ky)}{k^\sigma}$$

Proof. Since α just scales the time variable τ, we set $\alpha = 1$ without loss of generality.

For $\tau \leqslant 0$, the result was already shown in Section 4 to follow from Eq. (31) [it is also a direct consequence of the estimate (26)].

For $\tau > 0$, we choose $k < n/2$, and obtain an upper bound to $P_n(\tau)$ by relaxing all the constraints (36) for $k+1 \leqslant r \leqslant n-k-1$. We thus find the inequality

$$\begin{aligned} P_n(\tau) \leqslant \sqrt{2\pi n} \int_{-h_n(k)}^{\infty} dx_1 \int_{-h_n(k)}^{\infty} dx_2 \\ \times B_{n,k}(x_1)\, B_{n,k}(x_2) \frac{\exp[-(x_1 - x_2)^2/2(n-2k)]}{\sqrt{2\pi(n-2k)}} \end{aligned} \tag{38}$$

where $B_{n,k}$ denotes the conditional expectation

$$B_{n,k}(x) = E_{\mathrm{W}}[u(r) \geqslant -h_n(r),\, r = 1,..., k-1 \mid u(k) = x)] \tag{39}$$

To obtain (38) we have used the symmetry of the barrier (36) under the change $r \to n - r$. It follows from (39) that

$$P_n(\tau) \leqslant \sqrt{\frac{n}{n-2k}}\, (B_{n,k})^2 \tag{40}$$

where

$$B_{n,k} = \int_{-h_n(k)}^{\infty} dx\, B_{n,k}(x) = E_{\mathrm{W}}[u(r) \geqslant -h_n(r), r = 1,..., k] \tag{41}$$

is the Wiener measure of paths that are above the barrier $-h_n(r)$ at "times" $r = 1,..., k$. Since

$$\lim_{n \to \infty} h_n(r) = \tau r$$

one obtains by dominated convergence that

$$\lim_{n \to \infty} B_{n,k} = B_k = E_{\mathrm{W}}[u(r) \geqslant -\tau r, r = 1,..., k] \tag{42}$$

Moreover, the positive decreasing sequence B_k has a limit. Letting thus first $n \to \infty$ and then $k \to \infty$ in (40) leads to

$$\limsup_{n \to \infty} P_n(\tau) \leqslant B^2, \qquad B = \lim_{k \to \infty} B_k \tag{43}$$

To determine the quantity B we note that under the change $w(s) = u(s) + \tau s$,

$$B_k = E^{\tau}_{\mathrm{W}}[w(r) \geqslant 0, r = 1,..., k] \tag{44}$$

becomes the expectation (denoted by $E^{\tau}_{\mathrm{W}}[\cdot]$) for the random walk with shifted distribution of increments $\phi_\tau(y) = \phi(y - \tau)$. By the theorem of Sparre Andersen (see ref. 2, Section XII.7, Theorem 4), we know that the generating function

$$p(z) = 1 + \sum_{k=1}^{\infty} z^k B_k$$

of the B_k is given by

$$\log p(z) = \sum_{k=1}^{\infty} \frac{z^k}{k} E^{\tau}_{\mathrm{W}}[u_k \geqslant 0], \qquad 0 \leqslant z < 1 \tag{45}$$

Since the sequence B_k converges to B, we have also

$$\lim_{z \to 1} (1 - z)\, p(z) = B$$

and hence from (45)

$$\log B = \lim_{z \to 1} \log(1-z)\, p(z) = \sum_{k=1}^{\infty} \frac{1}{k} E_{\mathrm{W}}^{\tau}[u_k \leqslant 0] \tag{46}$$

Moreover, since

$$\begin{aligned} E_{\mathrm{W}}^{\tau}[u_k \leqslant 0] = E_{\mathrm{W}}[u_k \leqslant -\tau k] &= \frac{1}{\sqrt{2\pi k}} \int_{\tau k}^{\infty} dy \exp\left(\frac{-y^2}{2k}\right) \\ &= \sqrt{\frac{k}{2\pi}} \int_{\tau}^{\infty} dy \exp\left(\frac{-ky^2}{2}\right) \end{aligned} \tag{47}$$

one obtains

$$\log B = \sum_{k=1}^{\infty} \frac{1}{\sqrt{2\pi k}} \int_{\tau}^{\infty} dy \exp\left(\frac{-ky^2}{2}\right) \tag{48}$$

This shows that the upper bound B^2 in (43) is equal to the right-hand side of (37) for $\tau > 0$.

To obtain a lower bound we strengthen the constraints in (35) by the condition that Brownian paths remain above the point $h_n(k)$ for all "times" s belonging to the interval $[k, n-k]$. In this way one obtains

$$\begin{aligned} P_n(\tau) \geqslant \sqrt{2\pi n} \int_{-h_n(k)}^{\infty} dx_1 \int_{-h_n(k)}^{\infty} dx_2 \\ \times B_{n,k}(x_1)\, B_{n,k}(x_2)\, G_{h_n(k)}(x_1, k \,|\, x_2, n-k) \end{aligned} \tag{49}$$

where

$$\begin{aligned} &G_a(x_1, s_1 \,|\, x_2, s_2) \\ &\quad = \frac{1}{\sqrt{2\pi(x_2 - x_1)}} \left[\exp\left(-\frac{(x_2 - x_1)^2}{2(s_2 - s_1)} \right) - \exp\left(-\frac{(x_2 + x_1 - 2a)^2}{2(s_2 - s_1)} \right) \right] \end{aligned} \tag{50}$$

is the transition probability for Brownian paths restricted to the half-line $[a, \infty)$ with absorption at a.

It is shown in Appendix B that the right-hand side of inequality (49) tends also to B^2 when we choose $k = n^{\delta}$, $1/2 < \delta < 1$, and let $n \to \infty$. With (43) this concludes the proof of the proposition.

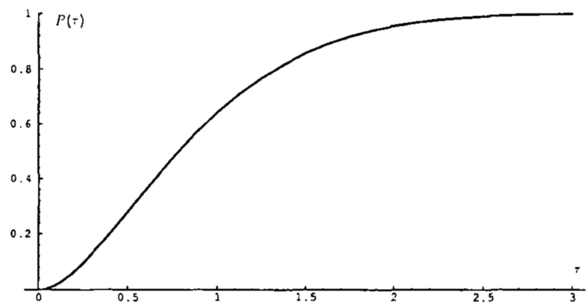

Fig. 1. Graph of probability $P(\tau)$ for $\alpha = 1$.

In the continuum limit the probability to form the total mass before t^* is identically zero. For $t > t^*$, the probability follows the law (37). In particular, its large-time behavior

$$P(\tau) \simeq 1 - 2 \int_{\alpha\tau}^{\infty} dy\, \phi(y), \qquad \tau \to \infty \tag{51}$$

agrees with that found in (29), where we let $n \to \infty$, $nm = M$ fixed. At $t = t^*$, we have $P(0) = 0$, and from the fact that[3]

$$F_{1/2}(y) \sim \sqrt{\frac{\pi}{y}}, \qquad y \to 0$$

we find

$$P(\tau) \simeq (\alpha\tau)^2, \qquad 0 < \tau \to 0 \tag{52}$$

The function $P(\tau)$ is shown in Fig. 1.

6. LAST COLLISION

In order to obtain some information about the ultimate stage of the gravitational collapse, we analyze the derivative

$$\frac{d}{d\tau} P_n(\tau) \geqslant 0 \tag{53}$$

The increase of $P_n(\tau)$ within the time interval $(\tau, \tau + \Delta\tau)$ is due to collisions. When $0 < \Delta\tau \to 0$, only the last collision contributes. Evaluating the derivative (53) thus yields information about the mass distribution just before complete aggregation.

From Eq. (24) we find that the derivative (53) is given by a sum of $(n-1)$ terms corresponding to crossing of the center-of-mass trajectories of clusters composed of k and $(n-k)$ particles ($k=1, 2,..., n-1$). The kth term has the structure

$$\frac{m}{2\lambda} k(n-k) \sqrt{2\pi n}\, \chi_k(\tau)\, \chi_{n-k}(\tau) \tag{54}$$

where

$$\begin{aligned}\chi_k(\tau) = \int du_1 \cdots \int du_{k-1}\, \phi(u_1)\, \phi(u_2-u_1) \cdots \phi(u_{k-1}-u_{k-2})\\ \times \phi\left(-\frac{m\tau}{2\lambda} k(n-k) - u_{n-1}\right)\\ \times \prod_{r=1}^{k-1} \theta\left\{u_r + \frac{m\tau}{2\lambda} r(n-r)\right\}\end{aligned} \tag{55}$$

Introducing new integration variables, one arrives at the simple result

$$\sqrt{2\pi k}\, \chi_k(\tau) = \exp\left\{-\frac{k}{2}\left[\frac{m\tau}{2\lambda}(n-k)\right]^2\right\} P_k(\tau) \tag{56}$$

We thus find the formula

$$\begin{aligned}\sqrt{2\pi n}\frac{d}{d\tau} P_n(\tau) = \frac{nm}{2\lambda} \sum_{k=1}^{n-1} \exp\left\{-\frac{nk(n-k)}{2}\left(\frac{m\tau}{2\lambda}\right)^2\right\}\\ \times \sqrt{k}\, P_k(\tau) \sqrt{n-k}\, P_{n-k}(\tau)\end{aligned} \tag{57}$$

At $\tau=0$, inserting relation (31), one gets

$$\begin{aligned}\sqrt{2\pi n}\frac{d}{d\tau} P_n(\tau) &= \frac{nm}{2\lambda} \sum_{k=1}^{n-1} \frac{1}{\sqrt{k(n-k)}}\\ &\simeq \frac{nm}{2\lambda}\pi, \qquad n \gg 1\end{aligned} \tag{58}$$

If we fix all parameters and let $n \to \infty$, the derivative diverges as $\sqrt{n}$, which is consistent with Eq. (33). On the contrary, in the physically relevant continuum limit, where $\alpha = nm/2\lambda$ is fixed, the rate of change of $P_n(\tau)$ at $\tau=0$ tends to zero. The corresponding asymptotic behavior of $P(\tau)$ is given in Eq. (52).

A very interesting conclusion follows from Eq. (57) in the region $\tau > 0$. For a macroscopic amount of initial matter ($n \to \infty$) the exponential factor

$$\exp\left\{-\frac{k}{2}\left(1-\frac{k}{n}\right)(\alpha\tau)^2\right\}$$

gives a nonvanishing weight only for a division of the system into a macroscopic cluster and a microscopic one: $k = \text{const}$, $(n-k) \to \infty$, or equivalently: $(n-k) = \text{const}$, $k \to \infty$. This means that at the ultimate stage of the evolution a macroscopic aggregate collides with a piece of a microscopic dust. In other words, in the continuum limit the essential aggregation can be expected to take place in the vicinity of time $t = t^*$, as for any $\tau > 0$ the last collision cannot involve two macroscopic masses. Further comments on this important point are given in the next section.

7. CONCLUDING REMARKS

In this paper we have provided a detailed analytic study of the probability $P_n(t)$ of forming the total mass $M = nm$ within time t. One should realize, however, that $P_n(t)$ describes only a particular aspect of the aggregation process: a complete description would be given by the determination of the evolution of the full mass spectrum in the course of time. The special feature of the information contained in $P_n(t)$ stems from the requirement that all collisions have taken place. The result (31) shows that before time t^* there is a vanishingly small probability of forming a single aggregate as $n \to \infty$. On the other hand, according to the discussion in Section 6, the dominant contribution to $P_n(t)$ for $t > t^*$ and $n \to \infty$ corresponds to the final aggregation of a microscopic mass km (k finite) with the preexisting macroscopic one $(n-k)\,m$. Thus the points of the curve of Fig. 1 represent events where the completion of the total mass occurs by final merging of some pieces of the leftover dust sent away from the center of mass of the system with a large initial momentum. These events are rare in the sense that they belong to a sampling of the tail of the initial velocity distribution. This motivates a less demanding question, namely to determine the probability of occurence of a macroscopic mass $\bar{M}$ within time t. By a macroscopic mass $\bar{M} = km$ we mean a finite fraction $\bar{M} = \eta M$ of the total mass, with $\eta = k/n$ fixed, $0 < \eta < 1$, as $n \to \infty$. In the continuum limit, preliminary computer simulations lead to the following picture, compatible with the above discussion:

(i) No macroscopic mass can be formed before t^*.

(ii) For any $t > t^*$, and $0 < \eta < 1$, a macroscopic mass is formed with probability one.

A plausible explanation for this phenomenon would be the rapid loss of kinetic energy during the first stage of collisions (short time scale of the order of a few times between collisions $\sim a/\lambda$). This initial stage is characterized by the occurence of many inelastic encounters between small masses causing an abrupt drop of the total kinetic energy, while no additional kinetic energy has been yet imparted to the bodies from the acceleration by gravitational forces. The subsequent evolution of the system is dominated by gravitation, so merging into a macroscopic mass occurs in the vicinity of the time t^*, as discussed in Section 3.

We comment now on the generalization of our results to larger classes of initial distributions. For the sake of obtaining explicit formulas, we have worked with the Gaussian probability density (22), but the whole analysis can be made for a general initial state with uncorrelated velocities where

$$\phi_\lambda(v)=\frac{1}{\lambda}\phi\left(\frac{v}{\lambda}\right)$$

with $\phi(v)$ the density of a continuous one-particle velocity distribution with zero mean and finite variance. A remarkable fact is that the relation (31) remains true for all such distributions. Considering the probability density $P_n(V, \tau)$ for the formation of the total aggregate $M=nm$ before time τ with velocity V, we would obtain the following generalization of the proposition in the continuum limit: setting $V=\sqrt{n}W$,

$$\begin{aligned}&\lim_{n\to\infty}\sqrt{n}\,P_n(\sqrt{n}\,W,\tau)\\&\quad=\frac{1}{\lambda\sqrt{2\pi}}\exp\left(-\frac{W^2}{2\lambda^2}\right)\exp\left\{-2\sum_{k=1}^{\infty}E_\phi[u_k\geqslant k\alpha\tau]\right\}\end{aligned} \tag{59}$$

In (59), E_ϕ is the expectation of the random walk u_k generated by ϕ. The distribution of the center-of-mass velocity decouples asymptotically acquireing Gaussian shape by the law of large numbers (in the Gaussian case the exact factorization occurs for all n).

If ϕ has support in an interval $[-v_{\max}, v_{\max}]$, the expectation $E_\phi(u_k\geqslant k\alpha\tau)$ vanishes whenever $\alpha\tau>2v_{\max}$, so the complete merging into a single mass must necessarily take place in the time interval $0<\tau\leqslant 4\lambda v_{\max}/M$, in accordance with the bound (19).

If the initial velocity distribution gives finite weight to discrete points [density $\phi(v)$ contains δ distributions], some care has to be exercised because the distinction between inequalities $\geqslant$ and strict inequalities $>$ in the constraints becomes relevant. The influence of the initial positional disorder is less clear. If we allow for small fluctuations around the regular lattice sites $x_j = ja$, $j = 1, \ldots, n$, the result will be qualitatively the same. But we have not been able to treat analytically random initial positions (Poisson statistics). It would be of interest to obtain more detailed information on the mass distribution in the course of time. In principle our method allows us to express such distributions in terms of the microscopic dynamics by writing down the appropriate constraints on the center of mass of the various clusters involved, but the formulas become cumbersome. We plan to come back to these questions both by analytic and numerical tools in future work.

The relevance of our model to astrophysics is questionable because it oversimplifies the real problem in two main aspects (see ref. 4, for instance). In one dimension, point particles surely hit each other, whereas (extended) bodies in higher dimensions may have low hitting probabilities. Moreover, the potential (1) is confining in contrast to the three-dimensional gravitational potential. A possible generalization of the one-dimensional dynamics, that could help remedy the first defect is to assume that, upon colliding, particles stick with probability q, $0 < q < 1$. However, replacing the potential (1) by a shorter range one will certainly spoil the mechanisms that make the present model solvable.

APPENDIX A

For $\tau > 0$, consider

$$B_k^\tau = E_W[u(r) \geqslant -\tau r,\ r = 1, \ldots, k] \tag{A.1}$$

the Wiener measure (42) of Brownian paths that are above the linear barrier $-\tau s$ for discrete times $s_r = r$, $r = 1, \ldots, k$, and

$$B_k(-\tau k) = E_W[u(r) \geqslant -\tau r,\ r = 1, \ldots, k-1 \mid u(k) = -\tau k] \tag{A.2}$$

the corresponding conditional expectation for paths that end in $-\tau k$ at time $s_k = k$. Notice that by the shift $w(s) = u(s) + \tau s$, we can also write

$$B_k^\tau = E_W^\tau[w(r) \geqslant 0,\ r = 1, \ldots, k] \tag{A.3}$$

$$B_k(-\tau k) = E_W^\tau[w(r) \geqslant 0,\ r = 1, \ldots, k-1 \mid w(k) = 0] \tag{A.4}$$

where E_W^τ is the expectation with respect to the shifted distribution $\phi_\tau(y) = \phi(y - \tau)$. From (A.4) the relation

$$B_k(-\tau k) = \exp\left(-\frac{k\tau^2}{2}\right) B_k(0) \tag{A.5}$$

follows. Differentiating (A.1) with respect to τ and using (A.5) yields

$$\frac{dB_k^\tau}{d\tau} = \sum_{j=1}^{k} j \exp\left(-j\frac{\tau^2}{2}\right) B_j(0)\, B_{k-j}^\tau \tag{A.6}$$

The monotonously decreasing sequence B_k^τ has a limit B. Since the sequences B_k^τ and $B_k(0)$ are bounded, the series (A.6) converges uniformly with respect to τ to

$$B \sum_{j=1}^{\infty} \exp\left(-j\frac{\tau^2}{2}\right) jB_j(0)$$

for $\tau > 0$. Thus, letting $k \to \infty$ in (A.6) yields

$$\frac{d}{d\tau} \log B = \sum_{j=1}^{\infty} \exp\left(-\frac{j\tau^2}{2}\right) jB_j(0) \tag{A.7}$$

On the other hand, an application of the theorem of Sparre Andersen [see Eqs. (44)–(48)] gives also the expression (48) for log B. Hence, differentiating (48) with respect to τ and comparing with (A.7) leads to

$$B_j(0) = \frac{1}{j\sqrt{2\pi j}} \tag{A.8}$$

from which the result (31) follows since $P_n(0) = (2\pi n)^{1/2} B_n(0)$.

APPENDIX B

Inserting (50) in (49), we decompose the lower bound as

$$P_n(\tau) \geqslant \sqrt{\frac{n}{n-2k}}\, B_{n,k}^2 - R_1(n,k) - R_2(n,k) \tag{B.1}$$

with

$$R_1(n,k) = \sqrt{\frac{n}{n-2k}} \int_{-h_n(k)}^{\infty} dx_1 \int_{-h_n(k)}^{\infty} dx_2\, B_{n,k}(x_1)\, B_{n,k}(x_2)$$
$$\times \left[1 - \exp\left(-\frac{(x_1 - x_2)^2}{2(n-2k)}\right)\right] \tag{B.2}$$

$$R_2(n,k)=\sqrt{\frac{n}{n-2k}}\int_{-h_n(k)}^{\infty}dx_1\int_{-h_n(k)}^{\infty}dx_2\,B_{n,k}(x_1)\,B_{n,k}(x_2)$$
$$\times\exp\left(-\frac{(x_1+x_2+2h_n(k))^2}{2(n-2k)}\right)\tag{B.3}$$

By removing the constraints in (39), we obtain

$$B_{n,k}(x)\leqslant\frac{1}{\sqrt{2\pi k}}\exp\left(-\frac{x^2}{2k}\right)$$

Hence

$$R_1(n,k)\leqslant\sqrt{\frac{n}{n-2k}}-1\tag{B.4}$$

and after a change of variables

$$R_2(n,k)\leqslant\sqrt{\frac{n}{n-2k}}\int dx_1\int dx_2\,\phi(x_1)\,\phi(x_2)$$
$$\times\exp\left[-\frac{[\sqrt{k}(x_1+x_2)+2h_n(k)]^2}{2(n-2k)}\right]\tag{B.5}$$

We set $k=n^\delta$ for some δ, $1/2<\delta<1$. Then $\lim_{n\to\infty}R_1(n,n^\delta)=0$, and we observe that the exponent in (B.5) is of the order $n^{2\delta-1}$ for fixed x_1, x_2. Thus $\lim_{n\to\infty}R_2(n,n^\delta)=0$ by dominated convergence.

It remains to estimate $B_{n,k}$. We have the inequality

$$B_k-B_{n,k}\leqslant\sum_{r=1}^{k}E_{\mathrm{W}}\left[-\tau r\leqslant u(r)\leqslant-\tau r\left(1-\frac{k}{n}\right)\right]$$
$$\leqslant\sum_{r=1}^{k}\frac{1}{\sqrt{2\pi r}}\int_{\tau r(1-k/n)}^{\tau r}dx\exp\left(-\frac{x^2}{2r}\right)$$
$$\leqslant\frac{k\tau}{n\sqrt{2\pi}}\sum_{r=1}^{k}\sqrt{r}\exp\left[-\frac{\tau^2 r}{2}\left(1-\frac{k}{n}\right)^2\right]=O\left(\frac{k}{n}\right)\tag{B.6}$$

Thus, for $k=n^\delta$, $1/2<\delta<1$, we obtain

$$\lim_{n\to\infty}B_{n,n^\delta}=\lim_{n\to\infty}B_{n^\delta}=B\tag{B.7}$$

Hence, setting $k = n^{\delta}$ in (B.1), and letting $n \to \infty$ gives

$$\liminf_{n \to \infty} P_n(\tau) \geqslant B^2$$

and this completes the proof of the proposition.

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